# CALCULATION AND MEASUREMENT OF THE TIME-CONSTANT OF A COOLING THERMOELEMENT IN THE STEADY STATE

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A simple interpolation formula is obtained for the time taken by a cooling thermoelement to reach a stationary condition, and an experimental check is carried out.

In the development of thermoelectric cooling devices, calculations must be made of not only their static parameters, but also dynamic parameters, and in particular the time for a thermobattery to reach a stationary condition. In most studies [1-3], in calculations of transient processes in a thermoelement, the authors have concentrated on the possibility of excess cooling of the working junction (compared with the stationary), in the initial period following the switching on of the current. A solution of the problem of a nonstationary temperature distribution in the branches of a thermoelement based on the heat conduction equation has a rather unwieldy appearance of a series of decaying exponentials. To determine their time-constants, the numerical solution of a transcendental equation is required. The problem is considerably simplified if only the time $\tau$ for establishing a stationary temperature fall $\Delta T_{st}$ is needed. For the main part of the time interval $\tau$, the thermoelement is in a steady-state thermal condition when in the series of exponentials for a nonstationary fall of temperature $\Delta T(t)$, there remains only one exponential with the smallest characteristic value $\lambda$:

$$\Delta T = \Delta T_{st}(1 - e^{-\lambda t}). \tag{1}$$

We define $\tau$ as the time in which $\Delta T$ reaches the value differing from $\Delta T_{st}$ by 5%. Since $e^{-3} \simeq 0.050$, it follows from (1) that $\tau = 3/\lambda$.

The time to establish a stationary fall in temperature is considered in [4] without solving the heat conduction equation by using a heat balance equation in the working junction, but it is not completely clear what is the basis and accuracy of the solution.

In the present work, a simple approximate solution is found, and checked experimentally, for the smallest characteristic value of the heat conduction equation. In accordance with the theory of a steady-state thermal condition [5], the time lag of a body in the steady state does not depend on heat sources, and is determined only by the dimensions of the body and the thermophysical parameters of the material. Therefore, in calculating the time-constant, as a first approximation the Peltier and Joule effects need not be considered, but a solution obtained of the simpler problem of temperature equalization in a nonuniformly heated thermoelement [6].

Let us place the origin of coordinates $x = 0$ on the heat-absorbing contact, and the heat-generating junction ($x = \ell$) will be considered as thermostated at temperature $T_0$ and the heat-absorbing one as adiabatically isolated, so that for temperature $T(x, t)$ we have the boundary condition at $x = 0$:

$$-\varkappa \frac{\partial T}{\partial x} + c_0 \frac{\partial T}{\partial t} = 0. \tag{2}$$

A solution of the heat conduction equation

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{a} \frac{\partial T}{\partial t} \tag{3}$$

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for the variable temperature component $\tilde{T}(x, t)$, will be looked for in the form [6]

$$\tilde{T} = T_c(x)\exp(-\lambda_c t). \tag{4}$$

From the heat conduction equation, and from boundary condition (2), we get for the values $\lambda_c$ a characteristic equation, which after introducing the dimensionless quantities

$$\delta_c = \sqrt{\frac{\lambda_c}{a}}\, l, \tag{5}$$

$$C = \frac{c_0}{c\gamma l} \tag{6}$$

takes the form

$$\operatorname{ctg}\delta_c = C\delta_c. \tag{7}$$

In the steady state we are interested in the least of the solutions of the characteristic equation, since it determines the temperature equalization time

$$\tau = \frac{3}{\lambda_1} = \frac{3l^2}{a}\frac{1}{\delta_1^2}. \tag{8}$$

Solutions of (7) are tabulated [7], but it is more convenient to use a simple approximation of the equation, which we obtain by considering the limiting cases $C \ll 1$ and $C \gg 1$, and keeping the two first terms in the expansion of $\delta_1$ in terms of a small parameter.

In the case $C \ll 1$, the value of $\delta_1$ is close to $\pi/2$ (and the second solution $\delta_2 \approx 3\pi/2$) and to a first-order approximation in C:

$$\delta_1 = \frac{\pi}{2}(1 - C), \quad \tau = 3\frac{l^2}{a}\left(\frac{4}{\pi^2} + \frac{8}{\pi^2}C\right). \tag{9}$$

For $C \gg 1$ the value of $\delta_1$ is small, $\delta_1 \ll 1$ ($\delta_2 \approx \pi$) and

$$\frac{1}{\delta_1^2} = \frac{1}{3} + C, \quad \tau = 3\frac{l^2}{a}\left(\frac{1}{3} + C\right). \tag{10}$$

Expressions (9) and (10) for $\tau$ are not much different from each other (approximately 20% in limiting cases). The interpolation formula

$$\tau = 3\frac{l^2}{a}\left(\frac{4}{\pi^2} + C\right) \tag{11}$$

is accurate in limiting cases, and in the worst case ($C \sim 1$) its accuracy is about 5%.

In all cases, the second solution $\lambda_2 \sim \delta_2^2$ is an order or more greater than $\lambda_1$, therefore, steady-state conditions are established in a time which is considerably less than the time to establish a stationary temperature, which justifies the use of the steady-state approximation in the case considered.

The temperature equalization time of the thermoelement obtained in the way adduced above coincides with the time for the thermoelement to reach a stationary condition after switching on a constant current density j, only if it is justifiable to assume that the intensity of the heat source, in particular the Peltier effect, is independent of temperature. Then, the calculated time-constant $\tau$, according to (11) turns out to be independent of current, while experimentally, a dependence $\tau(j)$ is observed. From this it follows that the approximation of temperature independence of the Peltier coefficient P is too rough, and Eq. (11) needs correction. We shall consider, as in [1], temperature independence of the thermal emf coefficient $\alpha = P/T$. Then in the boundary condition (2) there appears an additional term $\alpha jT$ and as a result of this, there is a term $(-\alpha j\ell/\kappa\delta_c)$ on the RHS of the characteristic equation (7). Considering the current to be small and expanding the solution $\delta_c$ in powers of j in the two limiting cases $C \ll 1$ and $C \gg 1$, with the same accuracy as we got the interpolation formula (11) above, we now find

$$\tau = 3\frac{l^2}{a}\left(1 + \frac{\alpha l}{\varkappa} j\right)^{-1}\left(\frac{4}{\pi^2} + \frac{c_0}{c\gamma l}\right). \tag{12}$$

The small parameter $\alpha\ell j/\kappa$ in which the expansion is made, is near to unity for a current optimal in the regime of greatest cold output, and usually smaller in the regime of greatest

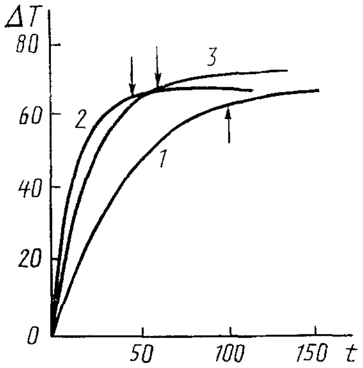

Fig. 1. Dependence of the temperature fall in thermoelectric cooling modules 1, 2, and 3, on the time elapsing after switching on the current. The arrows denote points on the curves at which the temperature fall had reached 0.95 of the steady-state value. ΔT in K; t in sec.

refrigeratory coefficient. In the experiments described below, this parameter varied in the range 0.25-0.70. We note that in the limiting case $C \gg 1$, which is closer to the experimental situation, than the contrary case, the fulfillment of both conditions $C \gg 1$ and $\alpha \ell j/\kappa \ll 1$ is not needed for getting Eq. (12). It is sufficient that the equality $C(1 + \alpha \ell j/\kappa)^{-1} \gg 1$ should be maintained, which is fulfilled for precisely those thermoelectric batteries described below in which $\alpha \ell j/\kappa$ has the greatest of the values quoted above.

Therefore, the use of Eq. (12) for calculating the time taken by a thermoelement or single-cascade thermobattery to reach a stationary condition is justified.

Let us compare Eq. (12) with results calculated by other methods. In solving the thermal conductivity equation by an operational method, the inversion of the Laplace transform T(s) to the original time function is easily accomplished if the denominator of the expression is expanded in powers of s. Thereby, we get Eq. (1) with time-constant described by Eq. (10), for $C \gg 1$, and its accuracy is determined by the difference from interpolation formula (11) (not more than 20%). As regards formulas obtained by Shcherbina [4] by solving the heat-balance equation on the heat-absorbing contact, then reducing it to a form analogous to (12), its only difference from (12) lies in the numerical term in brackets: 0.5 instead of $4/\pi^2 \simeq 0.4$, i.e., the accuracy of the approximate method is in the worst case 20%.

Let us comapre the computational formula (12) with experimental data.

Testing was carried out on three thermoelectric single-cascade modules manufactured from the same semiconducting materials, namely the solid solutions n-$Bi_2(Te_{0.9}Se_{0.1})_3$ and p-$(Bi_{0.25}Sb_{0.75})_2Te_3$. Constructional parameters of the modules are given in Table 1. On the heat-absorbing junctions, was a ceramic plate made of beryllium oxide.

Tests of the modules were carried out in the VUP-4 vacuum installation, at pressure no higher than $10^{-2}$ Pa. The modules, soldered to test plates, were on a thermostatted table at temperature 25 ± 0.5°C. Measurements of the temperatures of the heat-absorbing and heat-generating junctions were made with a type TKhA thermocouple. In Fig. 1 is shown the experimentally obtained temperature fall as a function of time from passing through the module the optimal current in the maximal temperature-fall mode. Measurements were also made with currents 0.7 and 0.5 of the optimal, which correspond to the optimal current in the greatest refrigeratory coefficient mode under the conditions in which there is a thermal load on the working contact. The time $\tau$ to get to a steady state was determined as that time to reach a temperature 0.95 of the stationary value. The steady-state temperature fall was determined at $t \simeq$ 10-15 min, when there was no further change in the recording instrument reading. In Table 2 are given the values of the currents, $\Delta T_{st}$ and experimentally obtained values of the time-constant in the steady-state regime.

The calculation of the $\tau$ values was made from Eq. (12) given above. In the calculation, we used the parameters of the semiconducting material [8]: $c = 0.15$ J/(g·K), $\gamma = 7.86$ g/cm³, $\kappa = 1.6$ W/(m·K), $\alpha = 195$ μV/K, and the parameters of the beryllium ceramic [9], the thermal capacity of which made up the main part of the thermal capacity of the contact plate, $c_c = 1.0$ J/(g·K), $\gamma_c = 3.1$ g/cm³. The value $c_0$, appearing in Eq. (12), was obtained from $c_0 = c_c \gamma_c S_c \ell_c / n\sigma$. Calculated values of $\tau$ are given in Table 2, together with experimental values. From a comparison of these, it is seen that Eq. (12) can be used to calculate $\tau$ with an accuracy of the order of 10-20%. Equation (12) also correctly describes the weak dependence of $\tau$ on the current magnitude.

TABLE 1. Parameters of the Thermoelectric Modules

| Module number | Number of branches | Length of branches, mm | Cross-section of branches, mm$^2$ | Area of ceramic plate, mm$^2$ | Thickness of ceramic plate, mm |
|---|---|---|---|---|---|
| 1 | 24 | 3,3 | 0,7×0,9 | 13×8 | 1 |
| 2 | 60 | 3,3 | 0,7×0,9 | 11×8 | 1 |
| 3 | 34 | 5,0 | 1,9×1,9 | 15×14 | 1 |

TABLE 2. Measured and Calculated Values of the Time-Constant

| Module number | Current, A | $\Delta T_{st}$, K | $\tau$, sec (exp.) | $\tau$, sec (calc.) |
|---|---|---|---|---|
| 1 | 1,10 | 68,0 | 98 | 83 |
|  | 0,77 | 63,4 | 106 | 95 |
|  | 0,55 | 54,6 | 112 | 105 |
| 2 | 1,00 | 68,2 | 38 | 33 |
|  | 0,70 | 63,3 | 38 | 37 |
|  | 0,50 | 54,6 | 43 | 41 |
| 3 | 2,90 | 72,2 | 53 | 41 |
|  | 2,03 | 65,2 | 58 | 45 |
|  | 1,45 | 56,0 | 62 | 49 |

During the calculation, it turned out that the thermal capacity of the contact plate influences the time lag of the miniature modules more than the thermal capacity of the semiconductor. In connection with this, it is interesting to compare modules 1 and 2, manufactured from identical semiconductor branches, and having almost identical ceramic plates, but differing widely in the times to reach a steady-state condition. The difference in $\tau$ is caused by the fact that module 2 contains considerably more branches than module 1 which affects $\tau$ through the value $c_0$, inversely proportional to n. In other words, in each thermoelement of module 2 there is a smaller volume of ceramic than in module 1. If, indeed, we express $\tau$ as a function of the quantities $C_c$ and K, characterizing the module as a whole, then the relative smallness of $\tau$ for module 2 is caused by the larger heat conduction of the thermal batteries consisting of a larger number of branches.

We can attempt to define more precisely the parameters used in the calculation. Taking into consideration the thermal capacity of the metallic layers of the contact plate, increases in $c_0$ are on the order of 30%. Another possible correction is for the temperature dependence of the parameters of the materials from which the modules are made. The specific thermal capacity of beryllium oxide depends especially strongly on temperature, falling by 1.8 times with a temperature lowering from room temperature to 200 K [9]. Therefore, with lowering of the temperature of the heat-absorbing contact during the transition process the thermal capacity of the beryllium ceramic gradually falls. As follows from the calculations made in the Appendix for the limiting case, when the thermal capacity of the semiconductor may be neglected in comparison with that of the contact plate, to make an approximate calculation of the temperature dependence of the thermal capacity of the latter, we must substitute in the equation for the time-constant the thermal capacity of the heat-absorbing contact at temperature $\bar{T} = T_0 - 0.68\Delta T_{st}$ averaged over the time interval $\tau$. Taking this into account reduces $c_0$ by 30-35%. So the corrections for the thermal capacity of the metal and for the temperature dependence of the thermal capacity of the ceramic therefore compensate each other rather well.

## APPENDIX

If the thermal capacity of the semiconductor is not taken into account then the temperature conductivity $a \to \infty$, and it is not necessary to solve the heat flow equation; it is sufficient to solve the heat balance equation for the heat-absorbing contact:

$$PI - \frac{1}{2}I^2R - K\Delta T - C_k\frac{d\Delta T}{dt} = 0. \qquad (A.1)$$

Taking the Peltier coefficient P, the resistance R, and the total heat conduction K of the module to be temperature-independent, and the total thermal capacity of the contact plate to depend linearly on temperature:

$$C_c(T) = C_0 - C_1\Delta T, \tag{A.2}$$

we get an equation for the quantity $y(t) = \Delta T_{st} - \Delta T(t)$:

$$Ky + (C_0 - C_1\Delta T_{st})\frac{dy}{dt} + C_1 y \frac{dy}{dt} = 0. \tag{A.3}$$

A solution of this nonlinear equation with initial condition $y = \Delta T_{st}$ $(t = 0)$ may be easily obtained in the form of the function $t(y)$:

$$t = -\frac{1}{K}\left[(C_0 - C_1\Delta T_{st})\ln\frac{y}{\Delta T_{st}} + C_1(y - \Delta T_{st})\right]. \tag{A.4}$$

In accordance with the definition of $\tau$, $t = \tau$ at $y = 0.5\Delta T_{st}$, and from (A.4), we get:

$$\tau = \frac{3C_c(\bar{T})}{K}. \tag{A.5}$$

A computation of the temperature of the heat absorbing contact averaged over the time interval from 0 to $\tau$, gives $\bar{T} = T_0 - 0.68\Delta T_{st}$ in the case of an exponential approximation of $\Delta T(t)$ to $\Delta T_{st}$, whence, from (A.2) and (A.5), we get

$$\tau = \frac{3}{K}(C_0 - C_1 \cdot 0.68\Delta T_{st}). \tag{A.6}$$

Thus, the time-constant $\tau$ is defined by the thermal capacity of the contact plate at temperature $\bar{T}$.

## NOTATION

$\tau$, time for the thermoelement to reach a stationary state; $\Delta T$, fall of temperature on the thermoelement: $\Delta T_{st}$, a stationary temperature fall; $\lambda$, characteristic value of the heat equation, see Eq. (1); t, time; x, coordinate; $\ell$, branch length of the thermoelement; T and $T_0$, temperatures of the heat absorbing and heat generating contacts, respectively; $\kappa$, heat conduction of the semiconductor; c, specific thermal capacity (of unit mass of the semiconductor); $\gamma$, density of the semiconductor; $a$, temperature conductivity; $c_0$, thermal capacity of commutation plate and cooled body referred to unit area of the transverse section of branches of the thermoelement; C, parameter defined by Eq. (6); $\delta_k$, quantity defined by Eq. (5); j, current density; $\alpha$, thermal emf; P, Peltier coefficient; s, Laplace transform variable; n, number of branches in the single-cascade thermobattery; $\sigma$, cross section of branches; $c_c$ and $\gamma_c$, specific thermal capacity and density of the contact plate; $S_c$ and $\ell_c$, area and thickness of the contact plate; I, current; R, resistance of the thermobattery; K, heat conduction of thermobattery; $C_c$, total thermal capacity of contact plate; y, temperature difference between the heat absorbing contact T and its stationary value; $\bar{T}$, temperature of the heat absorbing contact averaged over the time interval $\tau$.

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